PLANE ONE-DIMENSIONAL STATIONARY FLOW

OF AN IDEAL CHARGED GAS IN ITS OWN ELECTRIC FIELD

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The plane one-dimensional flow of an incompressible gas consisting of a neutral and a charged component in its own electric field has been investigated by Stuetzer [1]. Stuetzer's results are valid when the electrostatic pressure is small compared with the hydraulic pressure. In the present paper an analogous problem is considered for a compressible gas under the more general assumption that the pressures are comparable. Three cases are analyzed: a) the velocity of the relative motion of the charged and the neutral particles is equal to zero; b) it is nonvanishing but the flow can be assumed to be approximately isentropic; c) a nonisentropic flow, i.e., one cannot ignore irreversible losses due to the relative motion of the charged and neutral particles. In the first two cases, closed solutions are obtained.

1. We consider a plane flow, unbounded in the direction of the coordinates y and z, of a compressible gas along the ox axis (Fig. 1). Suppose that charges of one kind (assumed to be positive) are introduced in the gas flow in the yoz plane and are carried by the flow against the space-charge field to the section $y_-o_-z_-$ and are there neutralized. We shall call the yoz plane the emitter and the $y_-o_-z_-$ plane the collector. In writing down the system of equations we make the following assumptions:

$$\frac{\mu_0 V_0 H_0}{E_0} \ll 1, \quad \frac{n_{20}}{n_{10}}, \frac{\delta_{20}}{\delta_{10}} \ll \alpha_0 \lesssim 1 \quad \left(\alpha_0 = \frac{\varepsilon_0 E_0^2}{2p_0}\right) \tag{1.1}$$

(1.1)

$$f_{10}, f_{20} \ll \rho_0 E_0, \quad \frac{M_0^2}{R} \ll \alpha_0, \quad \frac{1}{(\gamma - 1) RP} \ll \alpha_0 \tag{1.2}$$

Here δ_i is the density of the i-th component (1 neutral, 2 charged); γ is the ratio of the specific heats; E_0 and H_0 are the characteristic strengths of the electric and the magnetic field; ε and μ are the permittivity and the magnetic permeability; n_i is the concentration of the i-th component; p is the hydrostatic pressure of the mixture of components; ρ is the electric charge density; P and R are the Prandtl and Reynolds numbers; V is the modulus of the mean velocity of the mixture of the components; f_1 and f_2 are the moduli of the forces $\mathbf{f}_1 = \mathbf{F}_1$, $\mathbf{f}_2 = \mathbf{F}_2 - \rho\mathbf{E}$, where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the volume densities of the sums of the forces acting on the neutral and charged components, respectively; the subscript zero indicates the characteristic values of the quantities.

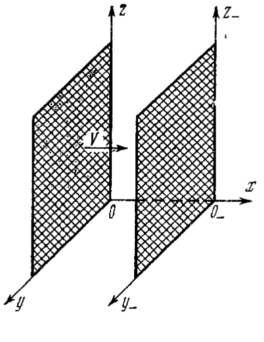

Fig. 1

The inequalities (1.1), which indicate that the acting magnetic forces are small compared with the electric forces (and then the particle velocity depends only on the time t and the spatial coordinates) and that the concentration of charged particles is small compared with that of the neutrals (in this case the contribution of long-range collisions is small),* enable one in this problem to

*As follows from Maxwell's equations of electrodynamics, $n_2/n_1 \sim \varepsilon E_0/eL_0 n_1$ (for atmospheric pressure, $E_0 \sim 10^7$ V/m, and $L_0 \sim 10^{-2}$ m, this gives $n_2/n_1 \sim 10^{-8}$) and the magnetic field produced by the motion of the charges is small and satisfies the first of the inequalities (1.1) if the characteristic velocity is small

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 24-32, January-February, 1971. Original article submitted March 20, 1970.

employ the equations of motion and expressions for the transport vectors obtained in the Chapman–Enskog theory of gaseous mixtures for a model of smooth hard spheres [2,3]. In accordance with (1.1), we take the concentration, density, pressure, temperature, and mean velocity of the mixture as a whole in these equations equal to the corresponding quantities of the neutral component, i.e., $n \approx n_1$, $\delta \approx \delta_1$, $p \approx p_1$, etc., and, taking into account the assumptions (1.2) (the last two of these determine the limits of applicability of the model of an ideal charged gas) we write the equations of conservation of the mass of each of the components and also the equation of conservation of the momentum and energy of the complete mixture in a plane one-dimensional stationary flow in the form

$$\frac{d}{dx}\delta V = 0, \quad \frac{d}{dx} j = 0 \tag{1.3}$$

$$\delta V \frac{dV}{dx} = \rho E - \frac{dp}{dx}, \quad \delta V \frac{d}{dx}\left(i + \frac{V^2}{2}\right) = jE \tag{1.4}$$

Here $i = c_pT$ is the enthalpy, j is the density of the electric current, and E is the electric field strength.

In accordance with [2, 3], the general expression for the current density vector can be written in the form

$$\mathbf{j} = \rho\langle\mathbf{w}_2\rangle = \rho\mathbf{V} + enD(\mathbf{d}_2 - k^T T^{-1}\operatorname{grad} T)$$
$$\mathbf{d}_2 = -\operatorname{grad}\frac{n_2}{n} - \left(\frac{n_2}{n} - \frac{\delta_2}{\delta}\right)\frac{1}{p}\operatorname{grad} p + \frac{1}{p\delta}(\delta_1\mathbf{F}_2 - \delta_2\mathbf{F}_1)$$

where D is the diffusion coefficient; e is the electric charge of a particle; k^T is the thermodiffusion ratio; $\mathbf{w}_2$ is the true velocity of a charged particle.

Using the well-known expression for k^T [2], estimating the orders of magnitude, and invoking (1.1) and (1.2), we obtain

$$k_0^T \lesssim \frac{\delta_{20}}{\delta_{10}}, \quad d_{20} \sim \frac{F_{20}}{p_0} \sim \frac{\rho_0 E_0}{p_0} \sim \frac{\varepsilon_0 E_0^2}{p_0 L_0} \sim \frac{\alpha_0}{L_0}$$
$$j_0 \sim \rho_0 V_0 + en_0 D_0\left(\frac{\alpha_0}{L_0} - \frac{1}{L_0}\frac{\delta_{20}}{\delta_{10}}\right) \sim \rho_0 V_0 + en_0 D_0 \frac{\alpha_0}{L_0} \sim \rho_0 V_0 + en_0 D_0 \frac{\rho_0 E_0}{p_0}$$

so that, setting $p = nkT$, we can write the expression for the current density in this case in the form

$$j = \rho(V + bE), \quad b = eD / kT \text{ is the mobility.} \tag{1.5}$$

To obtain a closed system, we must augment Eqs. (1.3)-(1.5) with the Clapeyron equation of state and Maxwell's equation for the electric field:

$$p = \delta R' T, \qquad dE / dx = \rho / \varepsilon \tag{1.6}$$

In Eqs. (1.3)-(1.6) we go over to dimensionless variables:

$$x' = x / L_0, \quad \delta' = \delta / \delta_0, \quad V' = V / V_0,$$

etc., and take the characteristic values of the quantities except the linear scale equal to the corresponding positive values of the variables at the emitter section, i.e., $V_0 = V_+$, $p_0 = p_+$, $E_0 = E_+ = |E(0)|$, etc., and the linear scale $L_0 = oo\_ = l$ such that at $x' = 0$

$$V' = 1, \quad p' = 1, \quad \delta' = 1, \quad T' = 1, \quad E' = -1 \tag{1.7}$$

We also assume that the mobility and the permittivity are constant, $b' = \varepsilon' = 1$. Then, applying the second of the relations (1.4) to the equation obtained by subtracting from it the first multiplied by the velocity and eliminating the temperature by means of (1.6) we write the original system thus:

$$\delta' V' = 1, \quad V'\frac{dV'}{dx'} = \frac{2\alpha_+}{\gamma M_+^2}\rho' V' E' - \frac{1}{\gamma M_+^2} V' \frac{dp'}{dx'} \tag{1.8}$$

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compared with the velocity of light $V_0^2/c^2 \ll 1$ and high-frequency processes are excluded, $t_0 V_0 / L_0 \gg V_0^2/c^2$ (L_0 is the characteristic length).

$$\frac{dp'}{p'}=\gamma\frac{d\delta'}{\delta'}+2(\gamma-1)\frac{\alpha_+}{R_e}\frac{\rho' E'^2}{p'}\frac{dx'}{V'} \tag{1.9}$$

$$j'=\rho'(V'+E'/R_e)=\text{const},\quad dE'/dx'=\rho' \tag{1.10}$$

Here R_e is the electric Reynolds number [4]. For what follows, we introduce the notation

$$a=\frac{{}^1\!/_2(\gamma-1)M_+^2}{1+{}^1\!/_2(\gamma-1)M_+^2},\quad I=j'2\left(\frac{\alpha_+}{\gamma M_+^2}\right)^{1/2}a^{3/4},\quad a_1=\frac{a^{(\gamma-1)/2}}{1+{}^1\!/_2(\gamma-1)M_+^2}$$
$$a_2=\left(\frac{\gamma M_+^2}{2\alpha_+}\right)^{1/2}\frac{a^{1/4}}{R_e},\quad a_3=\frac{4}{3}\frac{\alpha_+}{\gamma M_+^2}\frac{a}{R_e},\quad a_4=\frac{4\alpha_+}{\gamma M_+^2}a^{1/2} \tag{1.11}$$
$$a_5=\frac{1+\gamma M_+^2-\alpha_+}{{}^1\!/_2(\gamma+1)M_+^2}a^{1/2},\quad a_6=\frac{\alpha_+a^{1/2}}{{}^1\!/_2(\gamma+1)M_+^2a_4},\quad U'=-\int_0^{x'}E'\,dx'$$

Using the definition of the potential and eliminating from the second relation of (1.8) the quantities ρ', V', and p', we reduce the system of equations (1.8)-(1.10) to the single equation

$$\frac{1}{2}\left[\frac{a_2}{I}\frac{dW}{dx'}-\sqrt{2}I^2\left(\frac{d^2W}{dx'^2}\right)^{-1}\right]^2-\frac{1}{\sqrt{2}}\left[a_5+\frac{a_6}{I^2}\left(\frac{dW}{dx'}\right)^2\right]$$
$$\times\left[\frac{a_2}{I}\frac{dW}{dx'}-\sqrt{2}I^2\left(\frac{d^2W}{dx'^2}\right)^{-1}\right]+\frac{\gamma-1}{\gamma+1}(1-W)=0 \tag{1.12}$$

Here

$$W=\frac{jU}{\delta_+V_+({}^1\!/_2V_+^2+c_pT_+)}=\frac{2\alpha_+(\gamma-1)}{\gamma(1+{}^1\!/_2(\gamma-1)M_+^2)}j'U' \tag{1.13}$$

Equation (1.12) contains a single unknown function, $W(x')$. In accordance with (1.10) and (1.11), I is a constant and can be taken as a parameter.

For an isentropic flow, for which $(\gamma-1)\alpha_+/R_e \ll 1$ and, in accordance with (1.9), $p'=\delta'^{\gamma}$, we obtain a similar equation:

$$\frac{1}{2}\left[\frac{a_2}{I}\frac{dW}{dx'}-\sqrt{2}I^2\left(\frac{d^2W}{dx'^2}\right)^{-1}\right]^2+2^{1/2(\gamma-1)}a_1\left[\frac{a_2}{I}\frac{dW}{dx'}-\sqrt{2}I^2\left(\frac{d^2W}{dx'^2}\right)^{-1}\right]^{1-\gamma}+a_3-\frac{a_2}{3\sqrt{2}}\frac{1}{I^3}\left(\frac{dW}{dx'}\right)^3-(1-W)=0 \tag{1.14}$$

2. It is convenient to begin the solution of Eq. (1.14) by considering the case $R_e=\infty$ (the charged-particle mobility vanishes, $b=0$). We set $a_2=a_3=0$ and reduce the order of Eq. (1.14) by means of the substitution $f=(dW/dx')^2$ and we introduce the notation

$$y=-2I^2\left(\frac{df}{dW}\right)^{-1}=\left(a_1\frac{\gamma-1}{2}M^2\right)^{1/(\gamma+1)} \tag{2.1}$$

Then, the solution of Eq. (1.14) in parametric form can be written thus:

$$W=1-y^2-a_1y^{1-\gamma},\quad \varphi\equiv\frac{f}{I^2}=a_4+4(y-y_+)+2a_1\frac{\gamma-1}{\gamma}(y^{-\gamma}+y_+^{-\gamma}) \tag{2.2}$$

To find the relation between W and x', it is sufficient to determine the function $x'(y)$. The derivative of this function is

$$\frac{dx'}{dy}=-\left(\frac{dW}{dx'}\right)^{-1}\frac{y}{2I^2}\frac{df}{dy} \tag{2.3}$$

Now $dW/dx'=\pm f$, and, as follows from (1.13), the sign of dW/dx' is opposite to that of E'. In the emitter region, $E'<0$, and therefore $dW/dx'>0$. The field strength ($dE'/dx'=\rho'$) is a monotonically increasing function (we have assumed above that $\rho'>0$) and there therefore exists a section $x'=x_a'<1$, in the neighborhood of which E' changes sign. Taking this into account, eliminating the derivative dW/dx' from Eq. (2.3), and integrating the latter by parts, we find

$$Ix'=\begin{cases}y_+\sqrt{a_4}-y\sqrt{\varphi}+J_1(y_+,y) & (0\leqslant x'\leqslant x_a'),\quad J_1(\xi,\eta)=\int_\xi^\eta\sqrt{\varphi}\,dy\\ y_+\sqrt{a_4}+y\sqrt{\varphi}+J_1(y_+,y_a)-J_1(y_a,y) & (x_a'<x'\leqslant 1)\end{cases} \tag{2.4}$$
$$W=1-y^2-a_1y^{1-\gamma}$$

Here y_a is the value of y at the section $x' = x_a'$. In accordance with Eqs. (1.13) and (2.2), y_a can be found from the equation $\varphi(y_a) = 0$, which, using (2.1), we reduce to

$$\alpha_+ = 1 + \gamma M_+^2 - M_+^{\varkappa}(\gamma M_a^{\varkappa/\gamma} + M_a^{-\varkappa}) \quad (\varkappa = 2\gamma/(\gamma+1)) \tag{2.5}$$

Solutions of the transcendental equation (2.5) exist if

$$\alpha_+ \leqslant 1 + \gamma M_+^2 - (\gamma+1) M_+^{\varkappa} \tag{2.6}$$

At the same time, the solutions, of which there are two (M_{a1} and M_{a2}), satisfy

$$M_+ < M_{a1} < 1, \quad 1 < M_{a2} < M_+$$

The solutions are identical when there is equality in (2.6). We obtain a condition for the choice of the roots of Eq. (2.5) as follows. From the relations (1.8)

$$-\frac{\rho'}{\delta'} E'\, dx' = \frac{\gamma}{2\alpha_+} \delta'^{\gamma-1} \frac{dV'}{V'} (1 - M^2) \tag{2.7}$$

Since $(-\rho'/\delta')E'dx' > 0$ in the interval $0 \le x' < x_a'$, it follows from Eq. (2.7) that a subsonic flow is accelerated in this interval, whereas a supersonic flow is decelerated. Hence

$$M_a = M_{a1} \quad \text{for} \quad M_+ < 1, \quad M_a = M_{a2} \quad \text{for} \quad M_+ > 1$$

Let us consider the physical meaning of these roots; to this end we turn to Eqs. (2.4). In these equations, the dimensionless current density I, hitherto regarded as an arbitrary parameter, depends under otherwise equal conditions on the external electric resistance in the emitter-collector circuit. In (2.4) we set $x' = 1$ (collector section) and consider the function $W_-(I)$. (Here and below the subscript minus indicated quantities corresponding to $x' = 1$.) Its derivative is

$$\frac{dW_-}{dI} = \pm\sqrt{\varphi(M_-)}, \quad \text{if} \quad \frac{dW_-}{dx'}(M_-) \gtrless 0$$

and therefore the function $W_-(I)$ attains an extremum (maximum) when $\varphi(M_-) = 0$. At the same time, the relations (1.13), (2.2), and (2.5) show that the field strength at the collector vanishes, $E' = 0$, and $M = M_a$ and $x_a' = 1$.

We shall say that the case when $\varphi(M_-) = 0$ is optimal and denote the corresponding values of W_- and I by W_{opt} and I_{opt}.

If $I < I_{opt}$,

$$dW/dx' > 0 \quad (0 \leqslant x' \leqslant 1)$$

If $I > I_{opt}$,

$$dW/dx' \geqslant 0 \quad (0 \leqslant x' \leqslant x_a'), \quad dW/dx' < 0 \quad (x_a' < x' \leqslant 1)$$

Now $W_- < W_{opt}$ if $I \neq I_{opt}$ and, in accordance with Eqs. (2.4), $W_- = 0$ when $I = 0$ or $I = 2I_{opt}$. Using (2.1), this enables us to reduce Eqs. (2.4) to

$$Ix' = G\left[1 - \left(\frac{M_+}{M}\right)^{\varkappa'} \Phi(M) + J_2(M_+, M)\right] \tag{2.8}$$

$$W = 1 - \frac{M_+^{\varkappa'}}{1 + \frac{1}{2}(\gamma-1) M_+^2}\left(\frac{\gamma-1}{2} M^{4/(\gamma+1)} + M^{-\varkappa'}\right) \tag{2.9}$$

$$\gamma M_a^{\varkappa/\gamma} + M_a^{-\varkappa} = \frac{1 + \gamma M_+^2 - \alpha_+}{M_+^{\varkappa}} \tag{2.10}$$

Here

$$\Phi(M) = \left(\frac{M}{M_+}\right)^{\varkappa/2}\left[\frac{1 + \gamma M^2 - (\gamma M_a^{\varkappa/\gamma} + M_a^{-\varkappa}) M^{\varkappa}}{1 + \gamma M_+^2 - (\gamma M_a^{\varkappa/\gamma} + M_a^{-\varkappa}) M_+^{\varkappa}}\right]^{1/2}, \quad G = \frac{2a^{3/4}\alpha_+^{1/2}}{\gamma^{1/2} M_+}$$

$$J_2(M_+, M) = \frac{M_+^{-\varkappa''}}{\alpha_+^{1/2}} \int\limits_{M_+^{\varkappa/\gamma}}^{M^{\varkappa/\gamma}} \sqrt{\gamma z + z^{-\gamma} - (\gamma+1)}\, dz, \quad \varkappa' = 2\frac{\gamma-1}{\gamma+1}, \quad \varkappa'' = \frac{2-\gamma}{\gamma+1}$$

The system of equations (2.8)-(2.10) is valid if $0 \le I \le I_{opt}$, $0 \le x' \le 1$ or $I_{opt} \le I \le 2I_{opt}$, $0 \le x' \le x_a'$. If $I_{opt} \le I \le 2I_{opt}$, $x_a' \le x' \le 1$, Eq. (2.8) must be replaced by

$$Ix' = G\left[1 + \left(\frac{M_+}{M}\right)^{\varkappa'} \Phi(M) + J_2(M_+, M_a) - J_2(M_a, M)\right] \tag{2.11}$$

Equations (2.8)-(2.11) enable us to find the functions W(x') and M(x') if M_+, α_+, and I are given. We then determine the dependences U'(x'), E'(x'), and p'(x'), etc., by means of equations obtained from (1.10) and (1.13):

$$U' = \frac{W}{a_4^{1/2} I}, \quad E' = -\frac{1}{a_4^{1/2} I}\frac{dW}{dx'}, \quad \rho' = -\frac{1}{a_4^{1/2} I}\frac{d^2W}{dx'^2}, \quad \delta' = \left(\frac{M_+}{M}\right)^{\varkappa/\gamma} \tag{2.12}$$

and the relations (1.8), etc.

We obtain a system of equations for calculation of the optimal regime from (2.8)-(2.11) by setting $x' = 1$ and $M = M_- = M_a$:

$$I_{opt} = G[1 + J_2(M_+, M_-)]$$

$$W_{opt} = 1 - \frac{M_+^{\varkappa'}}{1 + {}^1/_2(\gamma-1)M_+^2}\left(\frac{\gamma-1}{2} M_-^{4/(\gamma+1)} + M_-^{-\varkappa'}\right) \tag{2.13}$$

$$\gamma M_-^{\varkappa/\gamma} + M_-^{-\varkappa} = \frac{1 + \gamma M_+^2 - \alpha_+}{M^{\varkappa}}$$

Consider the special case $\alpha_+ \ll 1$. In accordance with the last of Eqs. (2.13), the number M_- can, if we use Eqs. (2.13) with $\alpha_+ \ll 1$, be represented in the form $M_- = M_+(1 + z)$, where $|z| \ll 1$. Expanding expressions of the form $M^{\varkappa/\gamma}$, $M^{-\varkappa}$, etc., in a Taylor series, restricting ourselves to small terms of first order, and going over to the dimensionless quantities U and j, we find

$$j_{opt} = \frac{\varepsilon E_+ V_+}{l}, \quad U_{opt} = \frac{E_+ l}{2}, \quad W_{opt} = \frac{\gamma-1}{\gamma}\frac{\alpha_+}{1 + {}^1/_2(\gamma-1)M_+^2} \tag{2.14}$$

Similarly, setting $x' = 1$, in Eqs. (2.6), we obtain

$$\frac{U}{U_{opt}} + \frac{j}{j_{opt}} = 2$$

i.e., the current-voltage characteristic of the emitter-collector interval is a straight line if $\alpha_+ \ll 1$.

3. To determine the parameter $\alpha_+ = \varepsilon E_+^2/2p_+$, which occurs in Eqs. (2.8)-(2.11) and (2.13), we must know the field strength E_+ at the emitter section. In choosing the latter we have two possibilities: the current density j, and consequently E_+, is limited either by the emissive power of the emitter or the electric strength E_n of the medium (for example, when the emissive power is not restricted). In the first case, one can directly specify $E_+ < E_n$.

In the second case, the inequality $|E(x')| \le E_n(x')$ must hold at any section. Suppose E_n is proportional to the pressure:

$$\frac{E_n(x')}{E_{n+}} = \frac{p(x')}{p_+} \tag{3.1}$$

Then the equality $|E| = E_n$ will obtain at the section in which $|E/p|$ is maximal. Expressing this function by means of (1.11), (1.13), and (2.1) in terms of the number M and investigating when it has a maximum, we find that the maximum corresponds to the roots M_{b1} and M_{b2} of the equation

$$\alpha_+ = 1 + \gamma M_+^2 - M_+^{\varkappa}[(\gamma + {}^1/_2)M_b^{\varkappa/\gamma} + {}^1/_2 M_b^{-\varkappa}] \tag{3.2}$$

Comparison of this equation with (2.5) shows that $M_{b1} < M_{a1} < 1$ and $M_{b2} < M_{a2} > 1$. Now it follows from (2.7) that the flow velocity has an extremum at $M = M_a$ (a maximum for subsonic and minimum for supersonic flow); we conclude that a physical meaning attaches to only the root $M_{b1} < 1$. Obviously, M_{b1}

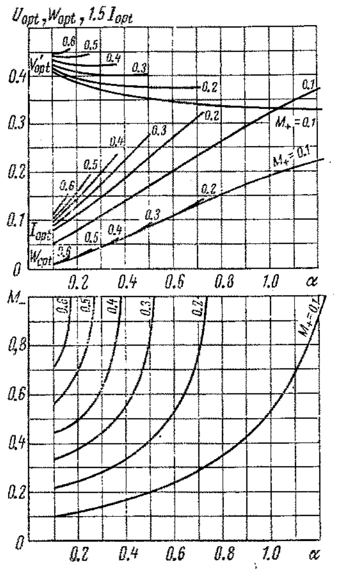

Fig. 2

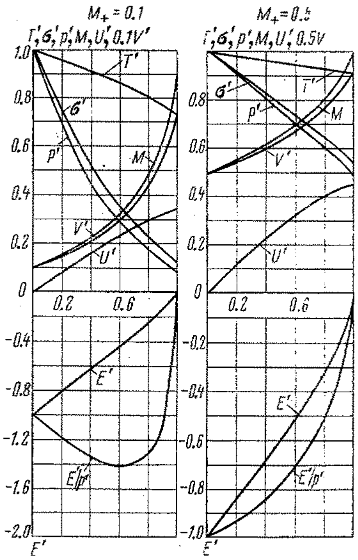

Fig. 3

can lie either within the interval $[M_+, M_a]$ or $[M_+, M_-]$ or without it if $M_{b1} < M_+$. In the first case, the maximal value of the function $|E/p|$ in the given interval corresponds to the section at which $M = M_{b1}$; in the second case, to the section at which $M = M_+$, i.e., the emitter plane. Using this result, expressing $|E/p|$ in terms of M, and using Eqs. (3.1) and (3.2), we obtain

$$\alpha_+ = \begin{cases} \alpha & (M_{b1} \leqslant M_+) \quad (\alpha = \varepsilon E_{n+}{}^2 / 2p_+) \\ \dfrac{2\alpha^2}{1 - M_{b1}{}^2} \left(\dfrac{M_+}{M_{b1}}\right)^{\varkappa} & (M_{b1} > M_+) \end{cases} \tag{3.3}$$

where M_{b1} is the smallest root on the interval [0, 1] of the equation

$$\frac{1 - M_b{}^2}{2}\left[(\gamma M_+^{\varkappa/\gamma} + M_+^{-\varkappa})\, M_b{}^{\varkappa} - \left(\gamma + \frac{1}{2}\right) M_b{}^2 - \frac{1}{2}\right] = \alpha^2$$

In contrast to α_+, the parameter α can be expressed in terms of the electric strength E_{n+} of the gas at the emitter plane, i.e., it can be assumed to be known.

Figure 2 shows W_{opt}, I_{opt}, U_{opt}', and M_- as functions of the parameter α for $M_+ < 1$, $\alpha > 0.1$, and $\gamma = 1.15$ obtained by solving Eqs. (2.13) with a computer; Fig. 3 shows U', E', etc., as functions of x' found in accordance with Eqs. (2.8)-(2.11) and (3.3) for the case $I = I_{opt}$, $\gamma = 1.15$. The graphs of the functions U'(x') and E'(x') for W = 0, $I = 2I_{opt}$, and $\alpha \ll 1$ are shown in Fig. 4.

4. Consider the case $R_e \neq \infty$ and $(\gamma - 1)\alpha_+/R_e \ll 1$. We use (2.2) and replace y by the variable

$$S = -2\left(\frac{d\varphi}{dW}\right)^{-1} \pm \frac{a_2}{\sqrt{2}}\, \varphi^{1/2}$$

In the same way as for $R_e = \infty$, we obtain

$$Ix' = \begin{cases} S_+\sqrt{a_4} - S\sqrt{\varphi} + J_1(S_+, S_a) - {}^1/_4 \sqrt{2}\,(a_4 - \varphi)\, a_2 & (0 \leqslant x' \leqslant x_a') \\ S_+\sqrt{a_4} + S\sqrt{\varphi} + J_1(S_+, S_a) - J_1(S_a, S) - {}^1/_4 \sqrt{2}(a_4 - \varphi) a_2 & (x_a' < x' \leqslant 1) \end{cases}$$

$$W = 1 - S^2 - a_1 S^{1-\gamma} - a_3 \pm {}^1/_6 \sqrt{2}\, a_2 \varphi^{3/2}, \quad \text{if} \quad x' \lessgtr x_a' \tag{4.1}$$

The investigation to establish when W_- has a maximum leads to the equations $S_- = S_a$, $\varphi(S_-) = 0$, and $E'_- = 0$, and these, in conjunction with Eqs. (2.10) and (4.1), yield

$$I_{opt}^* = I_{opt} - G / (2R_e), \quad W_{opt}^* = W_{opt} - {}^1/_3 G^2 a^{-1/2} R_e^{-1} \tag{4.2}$$

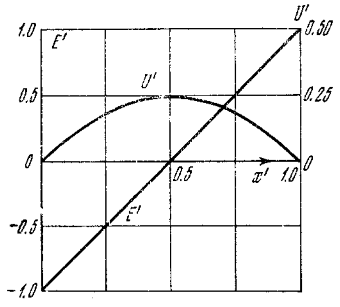

Fig. 4

Here $W_{opt}*$ and $I_{opt}*$ refer to the case $R_e \neq \infty$, and W_{opt} and I_{opt} refer as before to $R_e = \infty$. The relations (4.2) are convenient in that they make it possible to reduce the problem of finding $W_{opt}*$ and $I_{opt}*$ to the simpler problem when $R_e = \infty$. At the same time, if $1 < R_e < \infty$ (i.e., when $E_+ \leq E_{n+} < V_+/b$), the parameter α_+ can be found as in the case $R_e = \infty$, and if $R_e = 1$ (i.e., when $E_{n+} \geq E_+ = V_+/b$), then $\alpha_+ = \varepsilon V_+^2/2p_+b^2$.

In the special case $\alpha_+ \ll 1$ and $R_e = 1$, Eqs. (4.2) and (2.14) yield

$$j_{opt}^* = \frac{1}{2}\varepsilon\frac{V_+^2}{bl}, \quad U_{opt}^* = \frac{1}{3}\frac{V_+}{b}l, \quad W_{opt}^* = \frac{1}{3}\frac{\gamma-1}{\gamma}\frac{\alpha_+}{1+{}^1/_2(\gamma-1)M_+^2} \tag{4.3}$$

The first two of these expressions were first obtained by Stuetzer [1] in his study of the case V = const.

5. We now consider a nonisentropic flow. Reducing the order of the original equation (1.12) by means of the substitution $\psi = I^{-2}(dW/dx')^2$, solving it for the expression

$$\pm a_2\sqrt{\psi/2} - 2\,dW/d\psi,$$

extracting the square root, and taking the appropriate sign in accordance with the initial conditions, we obtain

$$\frac{dW}{d\psi} = \pm\frac{a_2}{2\sqrt{2}}\sqrt{\psi} - Y \tag{5.1}$$

Here

$$Y = \frac{a_5 + a_3\psi}{4} - \frac{1}{4}[(a_5 + a_6\psi)^2 - 2\varkappa'(1-W)]^{1/2}$$

and ψ lies on the interval $[0, a_4]$.

The solution of this equation cannot be found explicitly as in the case of an isentropic flow. For numerical integration, it is convenient to write Eq. (5.1) in such a form [see (1.11) and (1.13)].

If $\psi > 0$,

$$W = \int_{\psi}^{a_4}\left(Y - \frac{a_2}{2\sqrt{2}}\sqrt{\psi}\right)d\psi \tag{5.2}$$

If $\psi = 0$,

$$W = \int_0^{a_4}\left(Y - \frac{a_2}{2\sqrt{2}}\sqrt{\psi}\right)d\psi - \int_0^{\psi}\left(Y + \frac{a_2}{2\sqrt{2}}\sqrt{\psi}\right)d\psi \tag{5.3}$$

In the last expression, the first integral corresponds to the part of the emitter-collector gap where $E' \leq 0$ and the second to the part where $E' > 0$. Note that since $W \geq 0$, we have $\psi > a_4$ in the second interval. In particular, it follows from (5.2) and (5.3) that W_- is maximal at $\psi = 0$, i.e.,

$$W_{opt} = \int_0^{a_4}\left(Y - \frac{a^2}{2\sqrt{2}}\sqrt{\psi}\right)d\psi \tag{5.4}$$

As an example we give the results of calculations of W_{opt} for several values of R_e^{-1} for $\gamma = 1.1$ and two values of M_+. In the calculation, the values of α_+ were chosen such that $M_- = 1$ for $R_e^{-1} = 0$:

	$R_e^{-1} = 0$	0.2	0.4	0.6	0.8	1.0
$M_+ = 0.5$,	$W_{opt} = 0.154$	0.149	0.141	0.132	0.122	0.112
$M_+ = 0.1$,	$W_{opt} = 0.028$	0.024	0.021	0.018	0.015	0.012

6. The quantity W, which is the ratio of the generated electric energy to the total energy of the flow [see (1.13)], can be interpreted as the total efficiency of transformation. In accordance with the data given in Fig. 2 and the table, the efficiency for a fixed value of γ is larger, the larger are the parameter α and the number R_e (in practice, it is sufficient that $R_e \geqslant 10$). While the last condition can be satisfied if particles with low mobility are used as charge carriers [5], it is not easy to increase the value of the parameter α. For example, for air at atmospheric pressure, $\alpha \approx 5 \cdot 10^{-4}$, and the efficiency is $W \approx 10^{-4}$. For electronegative substances like CCl_4 [6], which have a high electric strength, we already have $\alpha \sim 0.15$ for $p_+ = 10^6$ N/m^2 and $W \approx 2.5 \cdot 10^{-2}$. A further increase of the efficiency is possible if one uses materials with a higher electric strength than CCl_4.

I should like to thank I. V. Bespalov and Yu. M. Trushin for their interest and helpful comments.

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